Line Integrals and Path Independence

We get to talk about integrals that are the areas under a line in three (or more) dimensional space. These are called, strangely enough, line integrals. Figure 11.1 shows an example of a curve in space, its "shadow" on the \( x, y \) plane, and the area between the curve and its shadow.

![Figure 11.1: A Line Integral](image)

Line integrals are written in a strange notation:

\[
\int_C [f(x,y)dx + g(x,y)dy], \tag{11.1}
\]

where \( C \) indicates that the integral is along a \textit{contour} or line, and \( f(x,y) \) and \( g(x,y) \) are functions of the independent variables \( x \) and \( y \). If there were three independent variables, equation 11.1 would have three terms.

Doing integrals of the form of equation 11.1 isn't really hard. In fact, it is often so easy that folks feel that they are doing something wrong. They aren't.

To begin with, you \textit{cannot} do a line integral unless you know the line or path along which the integral is to be done. A glance at Figure 11.1 will demonstrate that need. So in addition to the integral itself you also \textit{must} have an equation, the equation of the line under which the integral will be taken. This equation is known as the \textit{path}. The path is what makes line integrals easy. Line integrals only \textit{look} like they depend on two variables. They really don't.
They depend only on one. The *path* provides a relationship between the two variables, so they aren't both independent. The line us usually specified either by an equation, or sometimes by a set of equations. For instance let us evaluate the line integral

\[ I = \int_C [xy \, dx + x^2 \, dy], \quad (11.2) \]

along the line \( x = 2y \) from the point (2,1) to the point (4,2). We now know a starting point, an ending point, and the path connecting them. \( \text{11.6} \) The equation tells us that at every point at which we evaluate equation \( \text{11.2} \), \( x \) will be equal to \( 2y \). So I can replace \( x \) everywhere it occurs by \( 2y \), or, correspondingly \( y \) by \( x/2 \). Either way, I'm left with only one variable.

In fact, although I've not said it so far, integrals such as equation \( \text{11.1} \) or equation \( \text{11.2} \) can be broken into two parts:

\[ I = \int_C xy \, dx + \int_C x^2 \, dy. \quad (11.3) \]

And we can do different substitutions in each part. Let us remove the \( y \) in the first integral by replacing it with \( x/2 \). Then that integral depends only on \( x \). Too, we can get rid of the \( x^2 \) in the second integral by replacing \( x \) with \( 2y \). Then that integral will depend only on \( y \):

\[ I = \int_2^4 \frac{x^2}{2} \, dx + \int_1^2 4y^3 \, dy. \quad (11.4) \]

That's it! We are now left with two plain ordinary integrals to do. \( \text{11.7} \) They evaluate simply to 6 + 252 or 258.

It sometimes happens that one of the two terms in a line integral is missing. That doesn't matter, it is still a line integral. For instance

\[ I = \int_C y \, dx, \quad (11.5) \]

on the line \( y = e^x \) from (0,1) to (1,e) is evaluated just as the example above, but we've got only one term. So equation \( \text{11.5} \) becomes (making the obvious substitution)

\[ I = \int_0^1 e^x \, dx = e^x \bigg|_0^1 = e - 1. \quad (11.6) \]
Another point. Sometimes it is necessary to change the integration variable. This is done in the usual way. If, for instance, we were integrating

\[ I = \int_C x \, dy, \tag{11.7} \]

on the line \( x^2 = y \) from \((0,0)\) to \((1,1)\), we could get rid of the \( x \) by substitution, but we’d have to integrate \( y^{1/2} \, dy \). Instead we could simply change \( dy \) to \( 2x \, dx \) and integrate on \( x \) instead, giving \( I = 2/3 \) as the answer. \( 11.8 \)

Not all functions that look like total derivatives are total derivatives. There is no guarantee that any old line integral you think up will actually be the derivative of anything in particular. Those integrals depend on path. The ones that are total derivatives \( 11.24 \) do not depend on the path. Their value is always the same.

So we have something entirely new. There are two kinds of line integrals. Those that are independent of path and those that are not. Corresponding to them we have two kinds of differentials, those that are total differentials of some function and those that are not. The former have line integrals that are independent of path. The latter do not. In fact, the differentials that are total derivatives of some function are known as exact differentials. The others are known as inexact differentials.

**Exact and Inexact Differentials**

In a previous section we saw that certain line integrals were independent of the path of integration, while most line integrals are not. Indeed, we saw that the line integral from \(a\) to \(b\)

\[ I = \int_C \left[ f(x,y) \, dx + g(x,y) \, dy \right], \tag{12.1} \]

is independent of path if there is a function \( F(x,y) \) such that

\[ dF(x,y) = f(x,y) \, dx + g(x,y) \, dy. \tag{12.2} \]

In other words, the integral is independent of path if the integrand is the total derivative of some function. That statement needs thinking about. If the integrand in equation \( 12.1 \) is the total differential of some function, then it can be written:

\[ I = \int_C dF(x,y), \tag{12.3} \]

which integrates instantly into

\[ I = F(x,y) \bigg|_a^b, \tag{12.4} \]
which, of course, depends only on the endpoints \( a \) and \( b \) of the path. Differentials whose line integrals are independent of path are called exact differentials.

But, if the integrand in equation 12.1 is not the total differential of any function, then the simplification in equations 12.3 and 12.4 are not possible. And then to do the integral one must know the path. Such differentials are called inexact differentials.

The obvious question is "how do I tell if a given integrand is a total derivative or not?" One possibility is to simply try different functions until you either (a) find the parent function or (b) get very discouraged. A better way is to have a theory.

Let us take the total differential of a function \( F(x, y) \). This gives

\[
\left( \frac{\partial F}{\partial x} \right) y \ dx + \left( \frac{\partial F}{\partial y} \right) x \ dy = \frac{\partial F}{\partial x} \ dx + \frac{\partial F}{\partial y} \ dy.
\]

Now it is a curious fact that mixed second partial derivatives of a function, under very general conditions, are equal. In symbols that means:

\[
\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}.
\]

Or, in words, the order in which you differentiate \( F \) with respect to \( x \) and \( y \) isn’t important. Here’s an example: Let the function be

\[
F(x, y) = y^2 \ln(x) + \sin(xy)e^6 - x,
\]

then the two first partial derivatives are:

\[
\left( \frac{\partial F}{\partial x} \right) = \frac{y^2}{x} + y \cos(xy)e^{-x} - \sin(xy)e^{-x},
\]

\[
\left( \frac{\partial F}{\partial y} \right) = 2y \ln(x) + y \cos(xy)e^{-x}.
\]

Differentiating equation 12.8 with respect to \( y \)

\[
\left( \frac{\partial^2 F}{\partial y \partial x} \right) = \frac{2y}{x} + \cos(xy)e^{-x} - xy \sin(xy)e^{-x} - x \sin(xy)e^{-x},
\]

and equation 12.9 with respect to \( x \)

\[
\left( \frac{\partial^2 F}{\partial x \partial y} \right) = \frac{2y}{x} - \cos(xy)e^{-x} - xy \sin(xy)e^{-x} - x \cos(xy)e^{-x}.
\]
\[
\left( \frac{\partial^2 F}{\partial x \partial y} \right) = \frac{2y}{x} - \cos(xy)e^{-x} - xy \sin(xy)e^{-x} - x \cos(xy)e^{-x},
\]

which is identical to equation 12.10.

Now our problem is solved. If the integrand in equation 12.1 is the total differential of some function \( F(x, y) \), then it must be true that

\[
f(x, y) = \left( \frac{\partial F}{\partial x} \right), \tag{12.12}
\]

and

\[
g(x, y) = \left( \frac{\partial F}{\partial y} \right). \tag{12.13}
\]

And if these things are true, then the second mixed derivatives of \( F(x, y) \) must be equal. The second mixed derivatives are \( \partial f(x, y)/\partial y \) and \( \partial g(x, y)/\partial x \). So what we have found is that if, in a line integral such as equation 12.1,

\[
\left( \frac{\partial f(x, y)}{\partial y} \right) = \left( \frac{\partial g(x, y)}{\partial x} \right), \tag{12.14}
\]

then that line integral is independent of path.

What happens when there are more than two independent variables? The result is essentially the same. The mixed second derivatives of any function will be equal in pairs. For instance, the function \( F(x, y, z) \) has the following equal mixed second derivatives

\[
\left( \frac{\partial^2 F}{\partial x \partial y} \right) = \left( \frac{\partial^2 F}{\partial y \partial x} \right) = \left( \frac{\partial^2 F}{\partial x \partial z} \right) = \left( \frac{\partial^2 F}{\partial z \partial x} \right) = \left( \frac{\partial^2 F}{\partial y \partial z} \right) = \left( \frac{\partial^2 F}{\partial z \partial y} \right). \tag{12.15}
\]

For a three-variable line integral to be exact, all three equalities must hold.